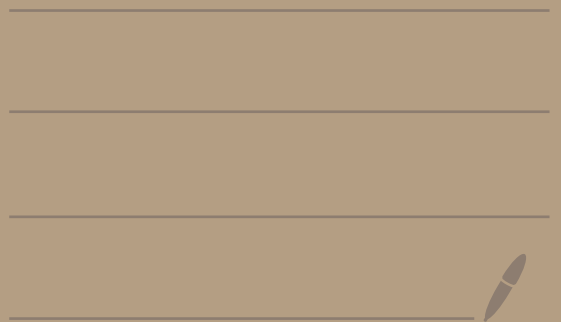


Topic 8 - Undetermined Coefficients



Topic 8 - Method of undetermined coefficients

We want to be able to solve:

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

Where a_2, a_1, a_0 are constants.

Method:

- ① Find general solution y_h to $a_2 y'' + a_1 y' + a_0 y = 0$ } topic 7
- ② Guess a particular solution y_p to $a_2 y'' + a_1 y' + a_0 y = b(x)$ } topic 8
- ③ The general solution to } use topic

$$a_2 y'' + a_1 y' + a_0 y = b(x)$$

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is

$$y = y_h + y_p$$

This table is on handout on the website. For guessing y_p .

$b(x)$	y_p guess for undetermined coefficients
constant	A
degree one polynomial such as: $5x - 3$ or $2x$	$Ax + B$
degree two polynomial such as: $10x^2 - x + 1$ or $x^2 + x$ or $2x^2 - 3$	$Ax^2 + Bx + C$
$\sin(kx)$ where k is a constant	$A \cos(kx) + B \sin(kx)$
$\cos(kx)$ where k is a constant	$A \cos(kx) + B \sin(kx)$
exponential such as: e^{kx} or $-2e^{kx}$	Ae^{kx}
degree one poly times exponential such as: xe^{kx} or $(2x + 1)e^{kx}$	$(Ax + B)e^{kx}$

Ex: Find the general solution to

$$y'' + 3y' + 2y = 2x^2$$

Step 1: Solve

$$y'' + 3y' + 2y = 0$$

The characteristic equation is

$$r^2 + 3r + 2 = 0$$

The roots are:

$$r = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{1}}{2}$$

$$= \frac{-3 \pm 1}{2} = \frac{-3+1}{2}, \frac{-3-1}{2} = \boxed{-1, -2}$$

Factor way:

$$(r+1)(r+2) = 0$$
$$r = -1, -2$$

We get

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2: Guess a solution y_p to

$$y'' + 3y' + 2y = 2x^2$$

Let's guess

$$y_p = Ax^2 + Bx + C$$

A, B, C
unknown
numbers

We have

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Plug these into the equation to get

$$\underbrace{(2A)}_{y_p''} + 3 \underbrace{(2Ax + B)}_{y_p'} + 2 \underbrace{(Ax^2 + Bx + C)}_{y_p} = 2x^2$$

We get

$$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = 2x^2$$

Regrouping:

$$\underbrace{2A}_2 x^2 + \underbrace{(6A+2B)}_0 x + \underbrace{(2A+3B+2C)}_0 = \underbrace{2x^2}_{2x^2+0x+0}$$

Get:

$$\begin{cases} 2A = 2 & \textcircled{1} \\ 6A + 2B = 0 & \textcircled{2} \\ 2A + 3B + 2C = 0 & \textcircled{3} \end{cases}$$

① gives $A = 1$.
Plug into ② and get $B = -3$

Plug into ③ and get $C = 7/2$

Thus,

$$y_p = Ax^2 + Bx + C = x^2 - 3x + \frac{7}{2}$$

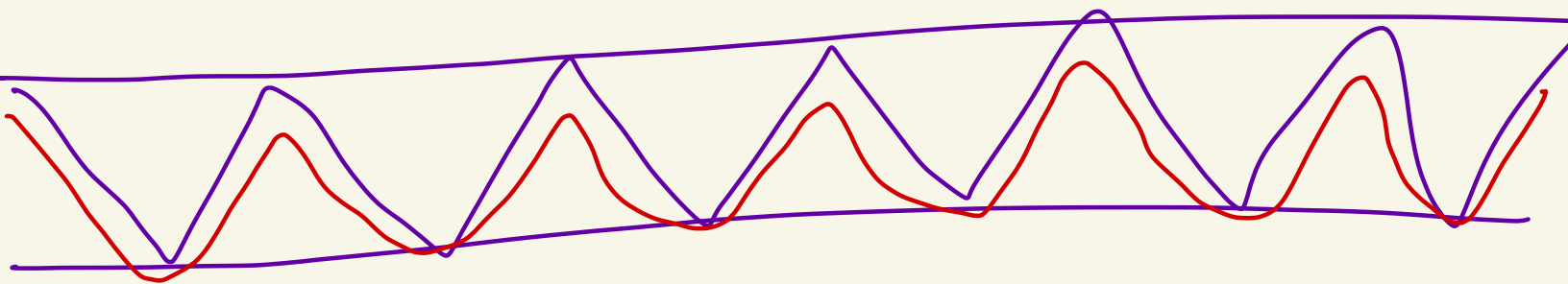
Step 3: The general solution to

$$y'' + 3y' + 2y = 2x^2$$

is

$$y = y_h + y_p$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + x^2 - 3x + \frac{7}{2}$$



Ex: Solve

$$y'' - y' + y = 2\sin(3x)$$

Step 1: Solve

$$y'' - y' + y = 0$$

The characteristic equation is

$$r^2 - r + 1 = 0$$

The roots are

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2} = \underbrace{\frac{1}{2} \pm \frac{\sqrt{3}}{2}i}_{\alpha \pm \beta i}$$

$$\alpha = 1/2, \beta = \sqrt{3}/2$$

General formula:

$$y_h = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

In our case we get:

$$y_h = c_1 e^{x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

Step 2: Guess a solution to

$$y'' - y' + y = \underbrace{2 \sin(3x)}_{b(x)}$$

Guess:

$$y_p = A \cos(3x) + B \sin(3x)$$

We get

$$y_p' = -3A \sin(3x) + 3B \cos(3x)$$

$$y_p'' = -9A \cos(3x) - 9B \sin(3x)$$

Plug these into the equation:

$$\begin{aligned}
 & (-9A \cos(3x) - 9B \sin(3x)) \\
 & - (-3A \sin(3x) + 3B \cos(3x)) \\
 & + (A \cos(3x) + B \sin(3x)) \quad \left. \begin{array}{l} y_p'' \\ - y_p' \\ + y_p \end{array} \right] \\
 & = 2 \sin(3x)
 \end{aligned}$$

Regrouping:

$$\begin{aligned}
 & \underbrace{(3A - 8B)}_2 \sin(3x) + \underbrace{(-8A - 3B)}_0 \cos(3x) \\
 & = 2 \sin(3x)
 \end{aligned}$$

Need

$$\begin{array}{l}
 \boxed{3A - 8B = 2} \quad (1) \\
 \boxed{-8A - 3B = 0} \quad (2)
 \end{array}$$

(2) gives $A = -\frac{3}{8}B$.

$$\text{Plug into (1): } 3 \underbrace{\left(-\frac{3}{8}B\right)}_A - 8B = 2$$

$$-\frac{9}{8}B - 8B = 2$$

$$-\frac{73}{8}B = 2$$

$$B = \frac{-16}{73}$$

$$\text{So, } A = -\frac{3}{8}B = -\frac{3}{8}\left(\frac{-16}{73}\right) = \boxed{\frac{6}{73}}$$

Thus,

$$y_p = \underbrace{\frac{6}{73}}_A \cos(3x) - \underbrace{\frac{16}{73}}_B \sin(3x)$$

Step 3: The general solution to

$$y'' - y' + y = 2 \sin(3x)$$

is

$$y = y_h + y_p$$

$$y = c_1 e^{x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{6}{73} \cos(3x) - \frac{16}{73} \sin(3x)$$

What can go wrong with the guessing method for y_p ?

If your y_p guess appears as a term in y_h then you need to multiply your guess by powers of x until your guess doesn't appear as a term in y_h

Ex: Solve

$$y'' - 5y' + 4y = 8e^x$$

Step 1: Solve

$$y'' - 5y' + 4y = 0$$

The characteristic equation is

$$r^2 - 5r + 4 = 0$$

$$(r-4)(r-1) = 0$$
$$r = 4, 1$$

The roots are

$$r = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2}$$

$$= \frac{5+3}{2}, \frac{5-3}{2} = \boxed{4, 1}$$

The solution to $y'' - 5y' + 4y = 0$

is $y_h = c_1 e^{4x} + c_2 e^x$

Step 2: Guess a solution y_p to

$$y'' - 5y' + 4y = \underbrace{8e^x}_{b(x)}$$

The table says to guess

$$y_p = Ae^x$$

This won't work, $y_p = Ae^x$ appears as a term in y_h .

Try plugging it into

We have $y_p = Ae^x$, $y_p' = Ae^x$, $y_p'' = Ae^x$

Plugging into the equation gives:

$$\underbrace{(Ae^x) - 5(Ae^x) + 4(Ae^x)}_{y'' - 5y' + 4y} = 8e^x$$

This gives

$$0 = 8e^x$$

This isn't solvable.

Since $y_p = Ae^x$ appears as a term in $y_h = c_1 e^{4x} + c_2 e^x$

we need to multiply our guess by an x .

Instead guess:

$$y_p = Ax e^x$$

We have $y_p' = Ae^x + Ax e^x$

$$\begin{aligned} y_p'' &= Ae^x + (Ae^x + Ax e^x) \\ &= 2Ae^x + Ax e^x \end{aligned}$$

Now plug into $y'' - 5y' + 4y = 8e^x$ to get:

$$\underbrace{(2Ae^x + Axe^x)}_{y_p''} - 5 \underbrace{(Ae^x + Axe^x)}_{y_p'} + 4 \underbrace{(Axe^x)}_{y_p} = 8e^x$$

This gives:

$$\underline{2Ae^x} + \underline{Axe^x} - \underline{5Ae^x} - \underline{5Axe^x} + \underline{4Axe^x} = 8e^x$$

0

We get:

$$-3Ae^x = 8e^x$$

Need

$$-3A = 8$$

$$\text{So, } A = -8/3$$

$$\text{Thus, } y_p = -\frac{8}{3}xe^x$$

$$\text{solves } y'' - 5y' + 4y = 8e^x$$

Step 3: The general solution to

$$y'' - 5y' + 4y = 8e^x$$

is

$$y = y_h + y_p$$

$$= c_1 e^{4x} + c_2 e^x - \frac{8}{3} x e^x$$

Ex: Solve

$$y'' - 2y' + y = e^x$$

Step 1: Solve

$$y'' - 2y' + y = 0$$

The characteristic equation is

$$r^2 - 2r + 1 = 0$$

We get

$$(r-1)(r-1) = 0$$

So we get a repeated
real root $r = 1$

Then,

$$y_h = c_1 e^x + c_2 x e^x$$

is the general solution to

$$y'' - 2y' + y = 0$$

Step 2: Now we guess y_p for

$$y'' - 2y' + y = \underbrace{e^x}_{b(x)}$$

The table says to guess $y_p = Ae^x$

But this appears in $y_h = c_1 e^x + c_2 x e^x$

So multiply by x and guess $y_p = Ax e^x$
our guess

But this appears in $y_h = c_1 e^x + c_2 x e^x$

So multiply our guess by x again

to get $y_p = Ax^2 e^x$

← doesn't appear in y_h

Now we plug it in.

$$y_p = Ax^2 e^x$$

$$y_p' = 2Ax e^x + Ax^2 e^x$$

$$\begin{aligned} y_p'' &= (2Ae^x + 2Ax e^x) + (2Ax e^x + Ax^2 e^x) \\ &= 2Ae^x + 4Ax e^x + Ax^2 e^x \end{aligned}$$

Plug these into

$$y'' - 2y' + y = e^x$$

to get:

$$(2Ae^x + 4Axe^x + Ax^2e^x) - 2(2Axe^x + Ax^2e^x) + Ax^2e^x = e^x$$

This gives:

$$\underbrace{2Ae^x}_{\text{orange}} + \underbrace{4Axe^x}_{\text{blue}} + \underbrace{Ax^2e^x}_{\text{pink}} - \underbrace{4Axe^x}_{\text{blue}} - \underbrace{2Ax^2e^x}_{\text{pink}} + \underbrace{Ax^2e^x}_{\text{pink}} = e^x$$

Diagram showing cancellation of terms with blue and pink lines, and a blue circle under the first $4Axe^x$ term and a pink circle under the first $2Ax^2e^x$ term.

We get:

$$2Ae^x = e^x$$

Need

$$2A = 1$$

So,

$$A = 1/2$$

Thus, $y_p = \frac{1}{2}x^2e^x$ solves

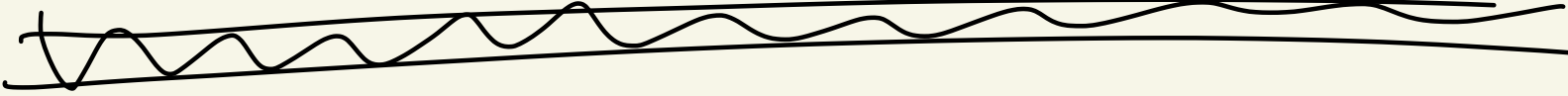
$$y'' - 2y' + y = e^x$$

Step 3: The general solution to

$$y'' - 2y' + y = e^x$$

is

$$y = y_h + y_p = c_1 e^x + c_2 x e^x + \frac{1}{2} x^2 e^x$$



There is also the situation where the $b(x)$ on the right side of the equation is a sum of terms. In this case your guess would be a sum of terms, one for each term in $b(x)$

Ex: For $y'' + 2y' = \underbrace{2x + 5}_{\text{you would guess:}} - \underbrace{e^x}_{\text{you would guess:}}$

$$y_p = \underbrace{Ax + B}_{\text{you would guess:}} + \underbrace{Ce^x}_{\text{you would guess:}}$$

Ex: For $y'' + y = \underbrace{2\sin(x)}_{\text{you would guess:}} + \underbrace{x^2}_{\text{you would guess:}}$

$$y_p = \underbrace{A\sin(x) + B\cos(x)}_{\text{you would guess:}} + \underbrace{Cx^2 + Dx + E}_{\text{you would guess:}}$$